

The Design of an Optimal Super-Twisting Differentiator under Measurement Noises

Marco Tulio Angulo*, Jaime A. Moreno and Leonid Fridman

Abstract—A method to compute the differentiation error in presence of bounded measurement noise for the family of Generalized Super-Twisting differentiators is presented. The proposed method allows choosing the optimal gain of each differentiator in the family providing the smallest ultimate bound of the differentiation error. In particular, an heuristic formula for the optimal gain of the pure Super-Twisting differentiator is presented. A numerical analysis comparing the performance of linear High-Gain and Super-Twisting differentiators is presented.

Index Terms—sliding-mode; differentiator; noise; optimal.

I. INTRODUCTION

The Super-Twisting (ST) algorithm has been successfully used in numerous applications. In particular, it has been widely used as a differentiator [1] in many contexts, see e.g. [2], [3], [4], [5] and the special issues [6] and [7].

The ST differentiator has two principal properties: exactness and robustness [1]. In the absence of noise, the ST differentiator is exact on the class of signals with bounded second derivative. No continuous differentiator can be exact on this class of signals. The second property is its robustness with respect to measurement noise. In the presence of measurement noise uniformly bounded by δ , its precision can be proportional to $\sqrt{L\delta}$, where L is the uniform bound of the second derivative of the signal. In addition, it was also shown that this is the best order of precision for any differentiator that is exact on this class of signals [1].

However, the analysis of the effect of measurement noise on the ST differentiator has only been qualitatively made, i.e., in terms of order of magnitude [1]. This means that the proportionality constant in its precision may be large or small depending on the selected gain. Moreover, there is no method to select the gain of the ST differentiator to improve its precision by minimizing this proportionality constant.

Our main contribution in this paper is to present a quantitative (and tight) analysis of the precision for the class of Generalized ST (GST) differentiators [8], [9]. The family of GST differentiators includes the ST and linear differentiators as particular cases. For a given pair (L, δ) , the analysis also allows computing the optimal gain for each differentiator in the class that minimizes the differentiation error. In particular, an heuristic formula for the optimal gain of the ST differentiator is given. Surprisingly, it turns out that the optimal gain of the ST differentiator does not depend on the noise amplitude.

The remainder of this paper is organized as follows. Section II formally introduces the Problem Statement. Section III presents the main results of the paper: a method to compute

the differentiation error in the presence of measurement noise. Section IV uses this method to compute the differentiation error for the pure ST differentiator. The proofs of all the Theorems of sections III and IV are collected in the Appendix. Section V presents the numerical results obtained from using the proposed method for linear, ST and GST differentiators. This section also presents an example where the pure ST differentiator outperforms the linear and GST differentiators and an heuristic formula to compute the optimal gain of the ST differentiator. Finally, some conclusions are given in Section VI.

II. PROBLEM STATEMENT

The problem consists in estimating the first derivative of a signal $\sigma(t)$ based on its noisy measurement $y(t) = \sigma(t) + \eta(t)$. Only two assumption will be made:

- (i) the second derivative of the base signal $\sigma(\cdot)$ is uniformly bounded by a known constant L ;
- (ii) the measurement noise $\eta(\cdot)$ is uniformly bounded by δ .

Setting $x_1 := \sigma$ and $x_2 := \dot{\sigma}$, the problem is transformed into the design of an observer for the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\rho, \quad y = x_1 + \eta, \quad (1)$$

based on the measured output y . In system (1), $\rho := -\ddot{\sigma}$ is a perturbation.

Let us consider the so-called Generalized ST differentiators [8], [9] in the following particular form

$$\dot{\hat{x}}_1 = -\frac{\alpha_1}{\varepsilon} \phi_1(\hat{x}_1 - y) + \hat{x}_2, \quad \dot{\hat{x}}_2 = -\frac{\alpha_2}{\varepsilon^2} \phi_2(\hat{x}_1 - y),$$

where $\alpha_i > 0$ are fixed¹ constants and $\varepsilon > 0$ sets the gain of the differentiator. The functions ϕ_1 and ϕ_2 are defined by

$$\begin{aligned} \phi_1(x) &= \mu_1 |x|^{\frac{1}{2}} \text{sign}(x) + \mu_2 x, \\ \phi_2(x) &= 0.5\mu_1^2 \text{sign}(x) + 1.5\mu_1\mu_2 |x|^{\frac{1}{2}} \text{sign}(x) + \mu_2^2 x, \end{aligned}$$

with $\mu_1, \mu_2 \geq 0$. The GST is reduced to a linear High-Gain differentiator when $\mu_1 = 0$ and to a pure ST differentiator when $\mu_2 = 0$.

By introducing the differentiation error $\tilde{x} = \hat{x} - x$ and defining

$$w_1 := \phi_1(\tilde{x}_1) - \phi_1(\tilde{x}_1 - \eta), \quad w_2 := \phi_2(\tilde{x}_1) - \phi_2(\tilde{x}_1 - \eta),$$

the dynamics of the differentiator error are given by

$$\dot{\tilde{x}}_1 = -\frac{\alpha_1}{\varepsilon} \phi_1(\tilde{x}_1) + \tilde{x}_2 + \frac{\alpha_1}{\varepsilon} w_1, \quad \dot{\tilde{x}}_2 = -\frac{\alpha_2}{\varepsilon^2} \phi_2(\tilde{x}_1) + \frac{\alpha_2}{\varepsilon^2} w_2 + \rho. \quad (2)$$

The problem consists in obtaining a tight estimate of the ultimate bound for the differentiation error \tilde{x}_2 in terms of the

¹A popular choice is $\alpha_1 = 1.5$, $\alpha_2 = 1.1$, originally given in [1].

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$$\mu_1 x^{1/2} + \mu_2 x = P_1 L \Gamma(x) \varepsilon^2 + Q_{12} \Gamma(x) [0.5 \mu_1^2 \Delta_0(x) + 1.5 \mu_1 \mu_2 \Delta_1(x) + \mu_2^2 \delta] + Q_{11} (\mu_1 \Delta_1(x) + \mu_2 \delta). \quad (3)$$

$$\Xi(x, \delta) = Q_{12} \Gamma(x) \frac{P_2}{P_1} [0.5 \mu_1^2 \Delta_0(x) + 1.5 \mu_1 \mu_2 \Delta_1(x) + \mu_2^2 \delta] + Q_{21} (\mu_1 \Delta_1(x) + \mu_2 \delta). \quad (4)$$

parameters of the differentiator and the bounds of the disturbances. In other words, the maximum asymptotic error that the differentiator will make due to the bounded disturbances. Once this expression is derived, the gain of the differentiator can be selected to improve its performance.

Remark about the Figures. In all figures and examples that follow, the parameters are set as $\alpha_1 = 1.5$, $\alpha_2 = 1.1$, $L = 1$, $\delta = 0.01$, unless otherwise stated.

III. MAIN RESULT

This paper extends the method of [10], valid strictly for linear time invariant systems, to compute the ultimate bound for the general nonlinear and discontinuous system (2).

Theorem 1: The ultimate bound for \tilde{x}_1 , denoted as $\tilde{x}_{1,ss}$, is the largest solution of equation (3) in the unknown x . Once this value is computed, the ultimate bound for \tilde{x}_2 is given by

$$\tilde{x}_{2,ss} = P_2 L \Gamma(x) \varepsilon + \frac{1}{\varepsilon} \Xi(x, \delta), \quad (5)$$

with $\Xi(x, \delta)$ defined as in expression (4).

Proof: See Appendix A. ■

In Theorem 1, the functions Δ_i and Γ are given by

$$\Delta_0(x) := 1 - \text{sign}(|x| - \delta), \quad \Gamma(x) := \frac{2|x|^{1/2}}{\mu_1 + 2\mu_2|x|^{1/2}},$$

$$\Delta_1(x) := |x|^{1/2} - ||x| - \delta|^{1/2} \text{sign}(|x| - \delta),$$

and note that, in fact, Δ_i is a function of x and δ . The constants P_i , Q_{ij} , $i, j = 1, 2$, are given² by

$$P_i = \int_0^\infty \{e^{At} B\}_i dt, \quad Q_{ij} = \int_0^\infty \{e^{At} D_j\}_i dt,$$

where the notation $\{\cdot\}_k$ denotes the k -th element of a two-dimensional vector. The matrices B , D_1 and D_2 are defined as

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D_1 = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ \alpha_2 \end{bmatrix}.$$

The ultimate bound (5) has two components, one depending on the perturbation L and the other depending on the noise δ . As the differentiator gain $1/\varepsilon$ increases the term due to the perturbation decreases but the noise term increases, and viceversa. This indicates a trade-off between the closeness to the true derivative in the absence of noise and the noise amplification in the presence of noise. This trade-off will be quantified in this paper for the GST differentiator (2), extending the quantitative results of [10] for the linear HG differentiator and the qualitative results of [1] for the ST differentiator. Note that the exact behavior of the ultimate bound $\tilde{x}_{2,ss}$ given by (5) is more complicate than it is apparent from the expression, since the value of x also depend on ε , L and δ . However, this qualitative trade-off is valid in general.

²For $\alpha_1 = 1.5$ and $\alpha_2 = 1.1$ they can be numerically evaluated using [11] giving $P_1 = 0.9852$, $Q_{11} = 1.3501$, $Q_{12} = 1.0838$, $P_2 = 1.5399$, $Q_{21} = 1.6257$, $Q_{22} = 1.6939$.

Geometrically, the solutions of (3) are the intersection points of the graphs of the two functions on the left and on the right of the equation (3), see Figure 1. In general, it is difficult to obtain analytical expressions for the maximal solution. However, it can be analytically solved at least in the pure ST case, as shown in the following section.

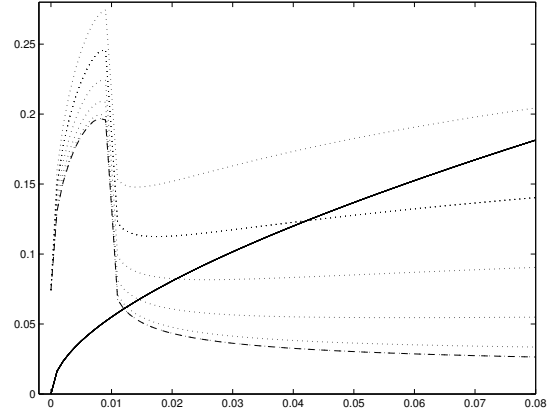


Fig. 1. Finding the solution of equation (3) of Theorem 1 is equivalent to find the intersection of two graphs. This gives the ultimate bound for \tilde{x}_1 . The solid line shows the graph of the left-hand side of (3). The dashed line shows the graph of the right-hand side of (3) for $\varepsilon = 0$; the dotted lines shows the graph of the same function for other values of $\varepsilon > 0$. The parameters $\mu_1 = \mu_2 = 0.5$ were used.

IV. ANALYSIS OF THE DIFFERENTIATION ERROR FOR THE ST DIFFERENTIATOR

When $\mu_2 = 0$, the equation (3) of Theorem 1 is reduced to

$$\left(\mu_1 - \frac{2\varepsilon^2 L P_1}{\mu_1}\right) x^{1/2} = Q_{11} \mu_1 \Delta_1(x) + \frac{1}{2} \mu_1^2 Q_{12} \Gamma(x) \Delta_0(x),$$

and the differentiation error to

$$\tilde{x}_{2,ss} = 2P_2 L \Gamma(x) \varepsilon + \frac{\mu_1}{\varepsilon} \left(\frac{P_2 Q_{12} \mu_1}{2P_1} \Gamma(x) \Delta_0(x) + Q_{21} \Delta_1(x) \right).$$

Theorem 2: The following statements are true:

- If the gain satisfies $\varepsilon^2 \leq \mu_1^2 / (2LP_1)$, there exists a finite ultimate bound for \tilde{x}_2 . Otherwise, the ultimate bound is “infinite”;
- the ultimate bound for \tilde{x}_1 is never smaller than δ ;
- $\tilde{x}_{2,ss} \rightarrow \infty$ either as $\varepsilon \rightarrow 0$, or as $\varepsilon \rightarrow \infty$.

Proof: See Appendix B. ■

This theorem shows that there exists a minimal value for the gain that guarantees that the ST differentiator is stable. If the gain is smaller than this value, the ST differentiator becomes unstable. Moreover, the theorem shows that it is impossible to make the ultimate bound for \tilde{x}_1 smaller than the amplitude of the noise, despite the selected gains. Finally, it shows that the differentiation error tends to infinite when the gain tends to zero or infinity. With this last observation, it is also possible

to conclude that there exists an optimal gain (or several of them) that provides the minimum differentiation error.

Theorem 3: When the gain is selected according to the amplitude of the perturbation using

$$\frac{1}{\varepsilon^2} = \frac{P_1 \theta}{\mu_1^2} L, \quad \theta > 2,$$

where θ is a new parameter, then the precision of the differentiator is of $O(\sqrt{\delta L})$. More precisely,

$$\tilde{x}_{2,ss} \leq \left(\frac{2P_2}{\sqrt{P_1 \theta}} \sqrt{c} + \sqrt{2P_1 \theta Q_{21}} \right) \sqrt{\delta L}$$

with $c = c(\theta)$ as displayed in formula (6).

Proof: See Appendix B. ■

The theorem shows that when the gain stabilizes and is selected proportional to $1/\sqrt{L}$, the precision is proportional to $\sqrt{\delta L}$. Both qualitative conclusions have been already obtained by Levant, see Theorem 2 of [1]. Theorem 3 improves those results in two aspects. Firstly, it provides the explicit value for the proportionality constants. Secondly, it does not require the assumption of “small enough noise”.

V. NUMERICAL EXAMPLES

Figure 2 presents the differentiation error as a function of the gain for the linear, ST and GST differentiators. The GST case is solved by numerically finding the solution of equation (3) using the *fzero* MATLAB function.

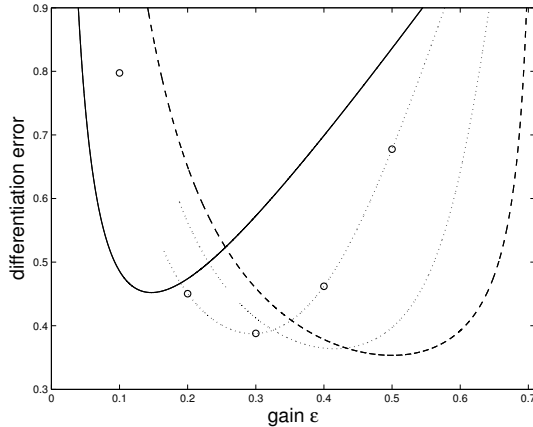


Fig. 2. Ultimate bound of the differentiation error as a function of ε , for $L = 1$ and $\delta = 0.01$. Solid: linear case $\mu_1 = 0, \mu_2 = 1$; dashed: pure ST case $\mu_1 = 1, \mu_2 = 0$; dotted: two experiments $\mu_1 = 0.8, \mu_2 = 0.2$ and $\mu_1 = 0.5, \mu_2 = 0.5$ (with circles).

The graph shows that the performance of the GST gets closer to the linear one as the gain of the ST part μ_1 tends to zero. Analogously, when $\mu_2 \rightarrow 0$ its performance gets closer to the ST one. In general, when $\mu_1 > 0$ and $\mu_2 > 0$, the performance of the GST is “between” the linear and ST in such a way that does not provide a smaller differentiation error than a pure ST differentiator.

The optimal gain ε^* for each differentiator can be found simply as the minimum of each curve. Note also that the graphs shows that the inclusion of linear terms to the ST differentiator avoids instability if the gain is not large enough. This is important when the actual value of L is not exactly known, as usual in practice.

A. Heuristic formula for the optimal gain of the ST differentiator.

Figure 2 also allows computing the optimal gain for a GST differentiator that provides the smallest ultimate bound of the differentiation error. For a given pair (L, δ) , the optimal gain can be computed by finding the ε at the minimum of the graph of $\tilde{x}_{2,ss}$.

Moreover, fixing the noise amplitude δ , one can obtain a graph of the optimal ε^* as a function of the perturbation amplitude L , as shown in Figure 3. The behavior of this graph can be described by the expression

$$\varepsilon^*(L) = \frac{m(\delta)}{\sqrt{L}} + b(\delta), \quad (7)$$

in congruence with the selection originally made by Levant in [1]. Equation (7) can be adjusted using only two values of the optimal gain $\varepsilon^*(L_1), \varepsilon^*(L_2)$ for two distinct values of the perturbation amplitude L_1 and L_2 . This yields

$$m = \frac{\varepsilon^*(L_2) - \varepsilon^*(L_1)}{1/\sqrt{L_2} - 1/\sqrt{L_1}}, \quad b = \varepsilon^*(L_1) - \frac{m}{\sqrt{L_1}}.$$

Several experiments have shown that formula (7) is correct in the sense that it can predict the correct values once it is adjusted. More surprising is the fact that this graph does not change when δ is changed! This means that m and b are indeed independent of δ , and need to be computed only once.

We have made this experiment obtaining the the graph shown in Figure 3 and the values³:

$$m = 0.4997 \approx 0.5, \quad b = 5.0494 \times 10^{-07} \approx 0, \quad (8)$$

which means that the optimal gain $1/\varepsilon^*$ of the ST differentiator is simply $2\sqrt{L}$, for all the amplitudes of noise.

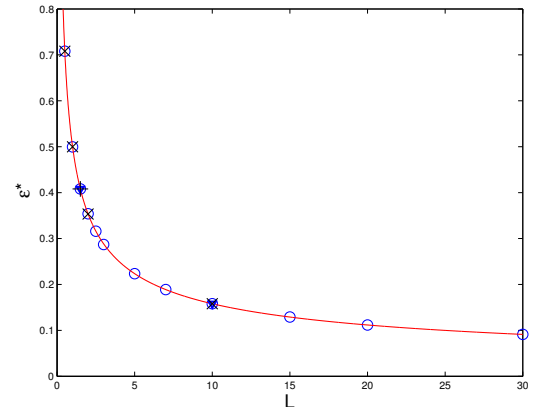


Fig. 3. Optimal ε of the pure ST differentiator as a function of L . The circles are measurement for $\delta = 0.01$, the cruxes for $\delta = 0.1$, the plus symbols for $\delta = 0.001$ and the triangles for $\delta = 0.05$. The interpolation using formula (7) adjusted using constants (8) is shown in solid line.

B. The performance of linear and ST differentiators.

In Figure 4 the graph of the differentiation error as a function of the gain is presented in two scenarios: when the perturbation is larger than the noise ($L > \delta$) and when it is smaller than the noise ($L < \delta$).

³These are the optimal values for $\alpha_1 = 1.5, \alpha_2 = 1.1$.

$$c(\theta) = \begin{cases} 1 & \text{if } Q_{11} < 1 \text{ and } \varepsilon^2 \leq \frac{\mu_1^2}{2LP_1}(1 - Q_{11}) \\ \frac{Q_{11}^2}{Q_{11}^2 - (\frac{2}{\theta} + Q_{11} - 1)^2} & \text{otherwise.} \end{cases} \quad (6)$$

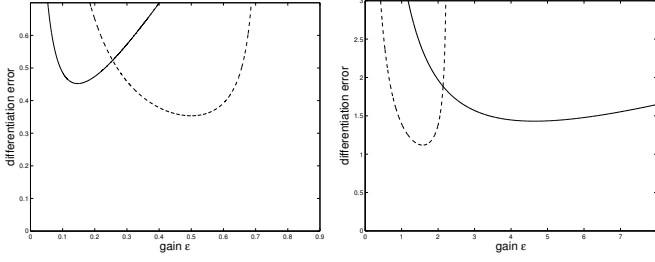


Fig. 4. Ultimate bound of the differentiation error as a function of ε . Solid: linear case $\mu_1 = 0, \mu_2 = 1$; dashed: pure ST case $\mu_1 = 1, \mu_2 = 0$. Left: parameters $L = 1, \delta = 0.01$; right: parameters $L = 0.1, \delta = 1$.

In this experiment, a ST differentiator provides a smaller error than a linear one in the two scenarios. This means that despite that the perturbation is small compared to the noise, one can select the gain of the ST small enough to obtain a smaller error than a linear differentiator.

Another important information one can recover from this experiment is the explicit value for the proportionality constant in the precision of a ST differentiator. It can be recovered by dividing the minimum value of the graph by $\sqrt{L\delta}$. This way, the proportionality constant for the ST precision is found to be 3.535. One can compare this value with 4.522, obtained for the precision of a linear differentiator.

VI. CONCLUSIONS

A method to compute the ultimate bound of the differentiator error for the family of GST differentiators was presented. The method allows obtaining the optimal gain of any GST differentiator and to compare the performance between them. In particular, we compared the minimum ultimate bound for the linear, ST and GST differentiators.

The pure ST differentiator can provide the minimum differentiation error. However, it is unstable when its gain is not large enough to overcome the perturbation. This can be alleviated by including small linear terms at the price of a slightly larger minimum error. Moreover, we heuristically obtained the optimal gain of the ST differentiator that, surprisingly, does not depend on the noise amplitude.

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APPENDIX

A. PROOF OF THE MAIN RESULT

The first step towards proving Theorem 1 is to transform the system into a more convenient form, as shown in the following result.

Theorem 4: The trajectory $\zeta(\cdot)$ of every solution of the system

$$\frac{d\zeta}{d\tau} = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon^2 \tilde{\rho} + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \tilde{w}, \quad \zeta(0) = \zeta_0, \quad (9)$$

with $\tilde{\rho} = \rho/\phi_1'$ and $\tilde{w} = (w_1, w_2/\phi_1')^T$ is transformed into a trajectory $\tilde{x}(\cdot)$ that is a solution to (2) using the formal change of coordinates

$$(\tilde{x}_1, \tilde{x}_2) = \phi^{-1}(\zeta) = \left(\phi_1^{-1}(\zeta_1), \frac{1}{\varepsilon} \zeta_2 \right).$$

Proof: Here we follow Filippov's ideas presented in [12, Chapter 2, pp. 99]. Let $\zeta(\tau)$ be a solution of (9), therefore, it is an absolutely continuous (AC) function. Set

$$t(\tau) = \varepsilon \int_0^\tau \frac{1}{\phi_1'(\tilde{x}_1(s))} ds,$$

where $\phi_1'(\tilde{x}_1) = 0.5\mu_1|\tilde{x}_1|^{-\frac{1}{2}} + \mu_2$.

Then the derivative $t'(\tau) = 2|\tilde{x}_1|^{1/2}/(\mu_1 + 2\mu_2|\tilde{x}_1|^{1/2})$ and assume, for the moment, that it is strictly positive, i.e., $\tilde{x}_1(t) \neq 0$. Therefore, there exists an inverse function $\tau(t)$, where $\tau'(t) = (1/\varepsilon)\phi_1'$. The function $\zeta^*(t) = \zeta(\tau(t))$ is AC, [12, Chapter 2, pp. 102], and

$$\frac{d\zeta^*}{dt} = \frac{d\zeta}{d\tau} \frac{d\tau}{dt} = \frac{d\zeta}{d\tau} \frac{1}{\varepsilon} \phi_1'$$

almost everywhere. Thus, the trajectory of any solution of

$$\frac{d\zeta}{dt} = \frac{1}{\varepsilon} \phi_1' \left(\begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon^2 \tilde{\rho} + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \tilde{w} \right), \quad (10)$$

is also the trajectory of some solution of (9), c.f., Theorem 3 of [12, Chapter 2].

Moreover, the coordinate transformation $\phi^{-1} : \zeta \mapsto \tilde{x}$ is one-to-one and class C^1 . The original differentiation error (2) is obtained by using this expression as a formal change of coordinates in system (10). Then, according to Theorem 1 of [12, Chapter 2], each solution of (10) is transformed into a solution of system (2).

Let us now consider what happens when $t'(\tau) = 0$. The function $2|\tilde{x}_1(t)|^{1/2}/(\mu_1 + 2\mu_2|\tilde{x}_1(t)|^{1/2})$ vanishes only in the points where the trajectory $\tilde{x}_1(t)$ is zero. When such points are isolated, the trajectory can be divided by such points into several (sometimes infinitely many) trajectories of the equation (10).

When the trajectory $\tilde{x}_1(s) \equiv 0$ for $s \in [a, b]$, the time t "stops". However, we will show that in such situation the trajectories of ζ and \tilde{x} in the phase space also stop. The condition $\tilde{x}_1(s) \equiv 0$ for $s \in [a, b]$, implies that $\zeta_1 \equiv 0$ on the same time interval. Using this fact on the system (9) yields

$$\zeta_2 = -\alpha_1 w_1, \quad \zeta_2 = \text{constant},$$

which means that the trajectory of (9) viewed in the phase space (ζ_1, ζ_2) stops. An analogous computation shows that the same occurs with the trajectory $\tilde{x}(t)$ of (2) when viewed in

the phase space $(\tilde{x}_1, \tilde{x}_2)$. Therefore, the relation $\tilde{x} = \phi^{-1}(\zeta)$ keeps being valid. ■

Due to lack of space, we state the following proposition without proof.

Proposition 1: The transformed disturbances satisfy

$$\begin{aligned} |\tilde{w}_1| &\leq \mu_1 \Delta_1(\tilde{x}_1) + \mu_2 \delta, \quad |\tilde{\rho}| \leq \Gamma(\tilde{x}_1)L, \\ |\tilde{w}_2| &\leq \Gamma(x_1) \left[\frac{1}{2} \mu_1^2 \Delta_0(\tilde{x}_1) + \frac{3}{2} \mu_1 \mu_2 \Delta_1(\tilde{x}_1) + \mu_2^2 \delta \right], \end{aligned} \quad (11)$$

where the functions Γ and Δ_i were introduced after the statement of Theorem 1.

Using the transformed system of Theorem 4 and the uniform bounds for the transformed disturbances $|\tilde{\rho}(t)| \leq \rho^0$, $|\tilde{w}_1(t)| \leq w_1^0$, $|\tilde{w}_2(t)| \leq w_2^0$, the ultimate bound for \tilde{x}_1 can be computed using [10, Lemma 1] as

$$\mu_1 \tilde{x}_{1,ss}^{1/2} + \mu_2 \tilde{x}_{1,ss} = P_1(\varepsilon^2 \rho^0 + \alpha_2 w_2^0) + Q_{11} w_1^0$$

and also for for \tilde{x}_2

$$\tilde{x}_{2,ss} = \frac{1}{\varepsilon} P_2(\varepsilon^2 \rho^0 + \alpha_2 w_2^0) + Q_{21} w_1^0.$$

A1. Proof of Theorem 1

In principle, [10, Lemma 1] could be directly used to compute the ultimate bound of the differentiation error based only on the uniform bounds of the disturbances. However, this would result in a very crude approximation of it, since the changes of the disturbances according to the state are ignored. To consider the dependance of the disturbances on the state, the following recursive algorithm is proposed:

[Step 0:] initialize the disturbances at its maximum, i.e., set

$$\begin{aligned} \tilde{\rho}^0 &= \frac{L}{\mu_2}, \quad \tilde{w}_1^0 = \mu_1 \sqrt{2\delta} + \mu_2 \delta, \\ \tilde{w}_2^0 &= \frac{1}{\mu_2} [\mu_1^2 + 1.5 \mu_1 \mu_2 \sqrt{2\delta} + \mu_2^2 \delta] \end{aligned}$$

[Step 1:] compute the corresponding ultimate bound for \tilde{x}_1 (here denoted x_0) as the unique solution to

$$\mu_1 x_0^{1/2} + \mu_2 x_0 = P_1 \varepsilon^2 \tilde{\rho}^0 + Q_{11} \tilde{w}_1^0 + Q_{12} \tilde{w}_2^0$$

[Step 2:] update the bounds of the disturbances using the obtained value of the ultimate bound

$$\begin{aligned} \tilde{\rho}^1 &= L\Gamma(x_0), \quad \tilde{w}_1^1 = \mu_1 \Delta_1(x_0) + \mu_2 \delta, \\ \tilde{w}_2^1 &= \Gamma(x_0) \left[\frac{1}{2} \mu_1^2 \Delta_0(x_0) + \frac{3}{2} \mu_1 \mu_2 \Delta_1(x_0) + \mu_2^2 \delta \right] \end{aligned}$$

[Step 3:] go back to Step 1 and repeat!

The algorithm defines two recursive maps:

$$v \mapsto x \text{ using } \mu_1 x^{1/2} + \mu_2 x = v$$

$$\begin{aligned} x \mapsto v \text{ using } v(x) &= P_1 L \varepsilon^2 \Gamma(x) + Q_{11} [\mu_1 \Delta_1(x) + \mu_2 \delta] + \\ &+ Q_{12} \Gamma(x) \left[\frac{1}{2} \mu_1^2 \Delta_0(x) + \frac{3}{2} \mu_1 \mu_2 \Delta_1(x) + \mu_2^2 \delta \right] \end{aligned}$$

We are now ready for the proof of Theorem 1:

Proof of Theorem 1: The argument is by induction. The first step of the algorithm obviously corresponds to an upper bound of the ultimate bound of the error. If at step n the ultimate bound is x_n and the disturbances are $(\tilde{\rho}^n, \tilde{w}_1^n, \tilde{w}_2^n)$, then the disturbances are indeed bounded by $(\tilde{\rho}^{n+1}, \tilde{w}_1^{n+1}, \tilde{w}_2^{n+1})$ so the ultimate bound is in fact x_{n+1} .

This shows that $x_n, n \geq 0$ are upper-bounds for the ultimate bound of \tilde{x}_1 . Then the limit of the algorithm (fixed point) is also an upper bound for the ultimate bound of \tilde{x}_1 .

Once the ultimate bound for \tilde{x}_1 has been computed, the ultimate bound for \tilde{x}_2 can be found by simply computing

$$\begin{aligned} \tilde{x}_{2,ss} &= \frac{1}{\varepsilon} \zeta_{2,ss} = \frac{1}{\varepsilon} \left[\frac{P_2}{P_1} (\mu_1 x^{*1/2} + \mu_2 x^*) + \right. \\ &\quad \left. + \left(Q_{21} - \frac{P_2}{P_1} Q_{11} \right) (\mu_1 \Delta_1(x^*) + \mu_2 \delta) \right] \end{aligned}$$

and by replacing $\mu_1(x^*)^{1/2} + \mu_2 x^* = v$ yields the claim of the Theorem. ■

B. PROOFS OF THEOREMS 2 AND 3

When $\mu_2 = 0$, the function $\Gamma(x)$ is reduced to $\Gamma(x) = (2/\mu_1)x^{1/2}$, and the intersection can be found by solving

$$\mu_1 x^{1/2} = \varepsilon^2 L P_1 \frac{2}{\mu_1} x^{1/2} + Q_{11} \mu_1 \Delta_1(x) + \frac{1}{2} \mu_1^2 Q_{12} \Gamma \Delta_0(x),$$

or, equivalently

$$\begin{aligned} v_0(x) &:= \left(\mu_1 - \frac{2\varepsilon^2 L P_1}{\mu_1} \right) x^{1/2} \\ &= Q_{11} \mu_1 \Delta_1(x) + \frac{1}{2} \mu_1^2 Q_{12} \Gamma \Delta_0(x) := v_1(x). \end{aligned}$$

Proof of Theorem 2:

a) Otherwise, $v_0(x) < 0$ and $v_1(x) \geq 0$, so there can not be an intersection. As $\varepsilon^2 \rightarrow \mu_1^2/(2LP_1)$, $v_0(x) \rightarrow 0$ uniformly, and the solution (intersection) grows, see Figure 5. When $\varepsilon^2 = \mu_1^2/(2LP_1)$, $v_0(x) \equiv 0$ and the only intersection (solution) is at infinity. When $\varepsilon^2 > \mu_1^2/(2LP_1)$ there is no solution and this should be interpreted as an infinite ultimate bound. In fact, this last conclusion has been proved using Lyapunov techniques: if the gain is not large enough, the differentiation error is unstable, c.f. [9].

b) The gain should be large enough to stabilize, i.e. $\varepsilon^2 \leq \mu_1^2/(2LP_1)$. We analyze two cases:

i) $Q_{11} < 1$. First note that $v_1(\delta^+) = Q_{11} \mu_1 \sqrt{\delta}$ and that

$$v_0(\delta) = \mu_1 \sqrt{\delta} - \frac{2\varepsilon^2 L P_1}{\mu_1} \sqrt{\delta}$$

then if $Q_{11} < 1$ we have that $v_1(\delta^+) < v_0(\delta)$ if

$$\varepsilon^2 \leq \frac{\mu_1^2}{2LP_1} (1 - Q_{11})$$

and for all those cases the intersection is at $x = \delta$. When $\varepsilon^2 > \frac{\mu_1^2}{2LP_1} (1 - Q_{11})$, but large enough to stabilize, then the intersection is with the second branch and is found by solving

$$\mu_1 x^{1/2} = \left(\frac{2LP_1}{\mu_1} \varepsilon^2 + Q_{11} \mu_1 \right) x^{1/2} - Q_{11} \mu_1 (x - \delta)^{1/2}$$

that yields

$$x = \frac{Q_{11}^2}{Q_{11}^2 - \left(\frac{2LP_1}{\mu_1} \varepsilon^2 + Q_{11} - 1 \right)^2} \delta. \quad (12)$$

Note that

$$x \rightarrow \frac{Q_{11}^2}{2Q_{11} - 1} \delta > \delta \text{ as } \varepsilon \rightarrow 0,$$

so the ultimate bound for \tilde{x}_1 cannot be smaller than δ , as claimed.

ii) $Q_{11} \geq 1$. In this case, $v_1(\delta^+) > v_0(\delta)$ so the intersection is only with the second branch of $v_1(x)$, see Figure 5. Therefore, the intersection always is at x obtained in point (i), equation (12), and it cannot be smaller than δ .

c) For any $\varepsilon > 0$, point (b) shows that $x \geq \delta$. In particular, x is larger than a positive value when $\varepsilon \rightarrow 0$. Therefore, $\Xi(x, \delta)$ tends to also a positive value when $\varepsilon \rightarrow 0$. Noticing that $\tilde{x}_{2,ss} \geq (1/\varepsilon)\Xi(x, \delta)$, one obtain that $\tilde{x}_{2,ss} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

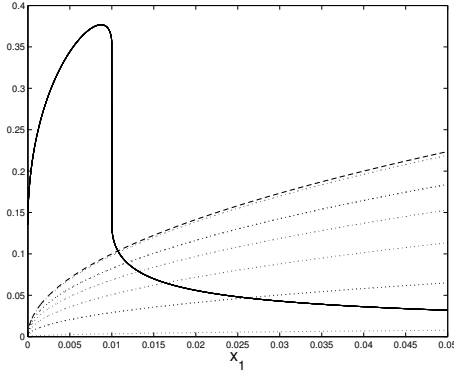


Fig. 5. Pure Super-Twisting case. Solid $v_1(x)$ for $\mu_1 = 1$ and $\delta = 0.01$. The dashed line is $v_0(x)$ for $\varepsilon = 0$, the dotted line is $v_0(x)$ as ε increases. In all the cases $L = 1$.

Proof of Theorem 3: If $\frac{1}{\varepsilon^2} = \frac{P_1 \theta}{\mu_1^2} L$ with $\theta > 2$, then there is an intersection since the following condition is satisfied

$$\varepsilon^2 \leq \frac{\mu_1^2}{2LP_1} \Leftrightarrow \frac{1}{\theta} < \frac{1}{2}$$

Therefore, the intersection is given by $x = \bar{c}\delta$, with $\bar{c} = \bar{c}(\theta)$ defined in equation (6).

Substituting the value for ε and using $\Delta_1(x) \leq \sqrt{2\delta}$ in the expression for $\tilde{x}_{2,ss}$ yields

$$\tilde{x}_{2,ss} \leq \left(\frac{2P_2}{\sqrt{P_1\theta}} \sqrt{\bar{c}} + \sqrt{2P_1\theta Q_{21}} \right) \sqrt{\delta L}$$

as claimed. ■

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